Evaluating the Maximum Allowable Drift in a Shear Wall with Variable Stiffness

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Abstract: - In a reinforced concrete building, it is a common knowledge that the estimation of the drift used by seismic codes as well as investigators in the field of wind analysis and earthquake engineering is based on experience and logic. None of the drift formulas used by seismic codes take into consideration the actual structural stiffness and concrete/steel properties. Maximum allowable drift ranges from $h/50$ in some codes to $h/2000$ in others, where $h$ is the height of a building. One of the main attempts to quantify maximum allowable drift was done by the author who suggested a formula that established grounds to start from in the estimation of the drift; it uses a constant lateral stiffness from the bottom to the top of a building. Shear walls however are usually designed with variable stiffness where the stiffer sections are at the bottom and the less stiff ones are as we go up the building. This paper analyses the building as a shear building, and uses a variable lateral stiffness between the stories for the lateral load resisting elements, and makes use of the finite element method along with structural dynamics and reinforced concrete design to generate a formula that can be used by a designer to estimate the allowable drift for a variable stiffness shear wall within elastic limits taking into consideration the effect of a cracked section suggested by UBC and ACI. In comparing results with other seismic codes, the suggested formula tends to be relatively conservative and close to the French code (PS92) and the Lebanese code.

Keywords: seismic codes, Finite Element Method, shear building, variable stiffness

1. INTRODUCTION

The maximum allowable drift suggested by various investigators give a value that ranges from $h/50$ to $h/2000$ [1-6]. This study makes use of the finite element method [7,8,9] along with structural dynamics [10] to evaluate the displacement at the top of a shear wall with variable stiffness and relates it to the maximum strain in the shear wall assuming that the maximum occurs in a balanced section at the point when tension reinforcing steel reaches the maximum allowable strain and when compression concrete crushes. This study then suggests an equation that can be followed in determining the maximum allowable drift; it is divided into five parts:

1- In the first part, the stiffness matrix of a shear wall is determined by making the assumption that a shear wall is a vertical beam in flexure with variable stiffness throughout its height.
2- In the second part, expressions for displacement and relative displacement at any level are determined.
3- In the third part, a general formula for the strain at the jth story along the shear wall is computed and maximum strain values are found.
4- In the fourth part, a combination of the results of the previous parts is used to obtain a relation between maximum displacement at the top of the shear wall and maximum strains at the bottom of every level of the shear wall. Consequently, an expression for the allowable drift at the top of the building is determined.
5- In the last part, an example is done using the suggested formula and a comparison between this formula and various selective seismic codes is done.

2. ASSEMBLY OF MATRICES (Shear Building)

As presented by Khouri [1], and after applying boundary conditions, the stiffness matrix of an element on the shear wall is:

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & -12 \\ -12 & 12 \end{bmatrix}$$

From Figure 1, and taking into consideration the boundary conditions presented in Khouri [1], matrix

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of eqn(1) below can be assembled relative to degrees of freedom 4, 7,…,j+1…3N+1. Also variable stiffness is assumed in all stories where $k^{(1)} \neq k^{(2)} \ldots \neq k^{(N)}$.

The stiffness matrix in this general case can be written as:

The Stiffness Matrix = $[A]$; \hspace{1cm} (1)

it can be represented in terms of the lateral stiffness of each story in the following manner:

$$
\begin{bmatrix}
1 & k^{(1)} & k^{(2)} & \ldots & k^{(N-1)} & k^{(N)} \\
k^{(1)} & 1 & k^{(1)+k^{(2)}} & \ldots & k^{(1)+k^{(N-1)}} & k^{(1)+k^{(N)}} \\
k^{(2)} & k^{(1)+k^{(2)}} & 1 & \ldots & k^{(2)+k^{(N-1)}} & k^{(2)+k^{(N)}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & k^{(1)}+k^{(2)}+\ldots+k^{(N-1)}+k^{(N)} \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
$$

Where, $k^{(i)}$ is the stiffness of one story,

$$
k^{(i)} = \sum_{j=1}^{n} \frac{E_j l_j^3}{12}.\]
Therefore, once the values of the displacements \( q' \) are known, the shear forces can be calculated.

Now, find matrix \( D \) as follows: \( A = B \times D \Rightarrow D = B^{-1} \times A \), and \( D = C^T \).

\[
D = \begin{bmatrix} k^1 & k^1 & k^1 & \cdots & k^1 \n k^1 & k^1 \times k^2 & k^1 \times k^2 & \cdots & k^1 \times k^2 \n k^1 & k^1 \times k^2 & k^1 \times k^2 & \cdots & k^1 \times k^2 \n \cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

\[
q = \frac{V}{k} \begin{bmatrix} (i-1) - \frac{(i-1)^2}{6X} 
 \vdots 
 (i-1) - \frac{(i-1)^2}{6X} 
 \vdots 
 (i-1) - \frac{(i-1)^2}{6X} 
 \end{bmatrix}\]

\[
q = \frac{V}{k} \begin{bmatrix} q_4 
 \vdots 
 q_{i-1} 
 \vdots 
 q_N = \Delta \end{bmatrix}
\]

where \( X = \sum_{i=1}^{N} \frac{N(N+1)}{2} \).

The displacement is calculated as: \( q' = D^{-1} \times q \) and can be presented as follows:

\[
\dot{q} = \frac{V}{\prod_{i=1}^{i=N} k^{(i)}} \sum_{i=1}^{i=N} \left[ \frac{(j-1)^2}{6X} \cdot k^{(i)} \right] \prod_{k=j}^{N} k^{(i)} \cdot \frac{(j-1)^2}{6X} \cdot \prod_{k=N}^{N} k^{(i)} 
\]

Now, compute the value of the relative displacement \( q_j' - q_{j-1}' \):

\[
q'_j - q'_{j-1} = \frac{V}{\prod_{i=1}^{i=N} k^{(i)}} \left[ \frac{(j-1)^2}{6X} \cdot k^{(i)} \right] \prod_{k=j}^{N} k^{(i)} \cdot \frac{(j-1)^2}{6X} \cdot \prod_{k=N}^{N} k^{(i)} 
\]

\[
q'_j - q'_{j-1} = \frac{V}{\prod_{i=1}^{i=N} k^{(i)}} \left[ \left( \prod_{k=j}^{N} k^{(i)} \right) \left( 1 - \frac{(j-1)^2}{6X} \right) + \left( 1 - \frac{(j-1)^2}{6X} \right) \prod_{k=N}^{N} k^{(i)} \right] 
\]

\[
q'_j - q'_{j-1} = \frac{V}{k^{(i)}} \left( 3j - \frac{3j^2}{2} + 6X \right) 
\]

F = A x q' => q' = A^(-1) x F, but A^(-1) = D^(-1) x B^(-1) =>

q' = D^(-1) x [B^(-1) x F].

The matrix \([B^{-1} F] = q\) represents the displacement in the particular case where the stiffness of all stories are the same as computed and presented in reference [1].
\[ q'_j - q'_{j-1} = \frac{V}{k^{(j)}} \left[ \frac{2X - j^2 + j}{2X} \right] \]  

(3)

4. COMPUTING MAXIMUM STRAIN

From reference [1], it was demonstrated that the strain in x direction along a shear wall between the two levels (j-1) and j is:

\[ \varepsilon_i = -\frac{y}{L^3} (12X - 6L) \left( q'_j - q'_{j-1} \right) \]  

(4)

Where,

\[ y \] is the algebraic distance measured from the neutral axis to the extreme fiber of shear wall section. \( y \) is considered to be positive in the opposite direction of the deflection \( U \). (Note that when \( y \) is replaced, \( x \) is the abscissa of the section along the shear wall between the two levels (j-1) and j.

\[ \varepsilon_x \] is the strain in x direction.

By substituting the values of \( (q'_j - q'_{j-1}) \) from eq.(3) into eq.(4), then

\[ \varepsilon_i = -\frac{y}{L^3} \left( 2X - j^2 + j \right) \left( \frac{V}{k^{(j)}} \right) \]  

(5)

Now, find the position of the maximum value of the strain \( \varepsilon_i \) between levels (i-1) and i.

\[ \frac{\partial}{\partial X} \left[ \varepsilon_i \right] = -12 \frac{y}{L^3} \left( \frac{2X - j^2 + j}{2X} \right) \left( \frac{V}{k^{(j)}} \right) < 0 \]

The function \( \varepsilon_i \) is decreasing which means that the maximum strain in a shear wall, between levels (j-1) and j, occurs at the bottom of the shear wall (x = 0), and this maximum strain is equal to:

\[ \varepsilon_{(x=0)} = \frac{6y}{L^2} \left( \frac{2X - j^2 + j}{2X} \right) \left( \frac{V}{k^{(j)}} \right) \]  

(6)

5. MAXIMUM DRIFT

The maximum drift at the top of the shear wall is reached when the strain in the reinforcement in the tensile zone at the critical section of the shear wall is equal to \( \varepsilon_{st} \) (maximum allowable strain in steel), and the strain in the extreme fiber of the compression zone in the same section is equal to \( \varepsilon_c = \) maximum strain limit of concrete in compression = 0.003. So the critical section in the shear wall is considered to have the behavior described in figure D.1.

Figure 2. Balanced Reinforced Concrete Section.

From eq(2),

\[ q'_j = \frac{V}{k^{(j)}} \sum_{k=1}^{N} \left[ \frac{\left[ 2X - j^2 + j \right]}{6X} \right] k^{(i)} - \left[ \frac{\left[ 2X - j^2 + j \right]}{6X} \right] k^{(j)} \]  

(8)

\[ \Rightarrow \]

\[ \Delta = q'_N = \frac{V}{k^{(j)}} \sum_{k=1}^{N} \left[ \frac{\left[ 2X - j^2 + j \right]}{6X} \right] k^{(i)} - \left[ \frac{\left[ 2X - j^2 + j \right]}{6X} \right] k^{(j)} \]  

(5)

Now, let

\[ R = \frac{1}{N} \sum_{k=1}^{N} \left[ \left[ \frac{\left[ 2X - j^2 + j \right]}{6X} \right] k^{(i)} - \left[ \frac{\left[ 2X - j^2 + j \right]}{6X} \right] k^{(j)} \right] \]  

But

\[ \Delta = R V \]  

(7)

Replace V in equation 6 =>

\[ \varepsilon_{(x=0)} = \frac{6y}{L^2} \left( \frac{2X - j^2 + j}{2X} \right) \left( \frac{\Delta}{R k^{(j)}} \right) \]  

(8)

where \( \Delta \) is the maximum displacement at the top of the shear wall. From similar triangles of the balanced section the following can be written:

\[ \frac{y}{\varepsilon_{st}} = \frac{x}{\varepsilon_c} \Rightarrow y = \frac{\varepsilon_{st}}{\varepsilon_c} \frac{d}{e} \]  

(9)

The maximum strain in the shear wall presented in eq(8) at the level of steel should be smaller than \( \varepsilon_y \) (the yield strain of steel):

\[ \varepsilon_{(x=0)} = \frac{6y}{L^2} \left( \frac{2X - j^2 + j}{2X} \right) \left( \frac{\Delta}{R k^{(j)}} \right) \leq \varepsilon_{st} \]  

(10)
Replace $y$ from eq(9) $\Rightarrow$

$$\Delta \leq \frac{2 \cdot R \cdot k^{(j)}(\varepsilon_{yd} + \varepsilon_{ec}) \cdot L^2}{6(2X - j^2 + j)d}$$  \hspace{1cm} (11)

$j$ ranging from 1 to $N$.

If the strain in the steel is considered to stay within $\varepsilon_y$ (the yielding strain of steel), the maximum allowable displacement at the top of the shear wall obeys the following equation,

$$\Delta \leq \frac{X \cdot R \cdot k^{(j)}(\varepsilon_y + \varepsilon_{ec}) \cdot L^2}{3(2X - j^2 + j)d}$$  \hspace{1cm} (12)

In this case, the critical section of the shear wall behaves as a balanced section; the limits are reached in the reinforcement in tension and in the concrete in compression at the same time, and at that point, the maximum allowable displacement at the top of the shear wall is reached.

Notice that as the height of a story increase, the maximum allowable displacement increases for a certain number of stories; and as $d$ increases, the maximum allowable displacement decreases since any small movement tends to cause larger strain at the critical section of the shear wall. On the other hand, and as far as maximum displacement is concerned and disregarding economical and architectural issues, it is better to use larger number of shear walls with small $d$ than to use fewer shear walls with large $d$; keeping in mind that the inertia of a shear wall is increased cubically as a function of $d$, and a bigger $d$ will increase our stiffness significantly in a certain direction.

6. MAXIMUM DRIFT CONSIDERING A CRACKED SECTION

It is important to note that according to UBC Modeling Requirement Section 1630.1.2 “Stiffness Properties of Reinforced Concrete and masonry elements shall consider the effects of cracked sections”.[1]

This means that when a designer uses cracked section analysis then according to ACI 318, section 10.10.4.1, the shear wall moment of Inertia would have to be reduced by a factor 0.35 such that the inertia would become 0.35$I_g$.[12] This will consequently reduce the stiffness of the lateral load resisting elements and will equally increase the drift by the same factor, which means that the drift should be multiplied by a factor of $(1/0.35)$ to get a value taking into consideration the effects of a cracked section suggested by UBC-97 and ACI-318-08.

$$\Delta m \leq \left(\frac{1}{0.35}\right) \frac{2 \cdot R \cdot k^{(j)}(\varepsilon_{yd} + \varepsilon_{ec}) \cdot L^2}{6(2X - j^2 + j)d}$$  \hspace{1cm} (13)

7. EXAMPLE

In order to see how this formula works, let’s consider an example of a 20 story building with shear walls having variable lateral stiffness as we go up the building.

Given that for a 20 story building, the lateral load resisting stiffness varies as follows:
- from level 1 to 5 the stiffness is k,
- from level 6 to 10 the stiffness is 0.9 k,
- from level 11 to 15 the stiffness is 0.8k,
- and from level 16 to 20 the stiffness is 0.7k.

Yield Strength, $f_y = 414$ Mpa, Modulus of Elasticity, $E_s = 2.10^5$ Mpa, Reinforcement Steel Yield Strain, $\varepsilon_y = 0.00207$, Story Height = $L = 3m$; Concrete Crushing Strain $\varepsilon_c = 0.003$.

* The shear walls have a constant width $b$.
* The biggest shear wall have $h = 4m$ at the base, so the depth $d = 0.9h = 3.6m$ at the base and diminishes when moving upwards.

The results of the suggested equation are presented in Table 1.

Table 1. Allowable drift for Example 1 using Eqn (13).

<table>
<thead>
<tr>
<th>Story</th>
<th>Stiffness</th>
<th>Displacement</th>
<th>Strain, $\varepsilon$</th>
<th>$\Delta$</th>
<th>$\Delta m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 k</td>
<td>1.000 V/k</td>
<td>0.667</td>
<td>$y^*V/k$</td>
<td>0.0327</td>
</tr>
<tr>
<td>2</td>
<td>1 k</td>
<td>1.995 V/k</td>
<td>0.663</td>
<td>$y^*V/k$</td>
<td>0.0329</td>
</tr>
<tr>
<td>3</td>
<td>1 k</td>
<td>2.981 V/k</td>
<td>0.657</td>
<td>$y^*V/k$</td>
<td>0.0332</td>
</tr>
<tr>
<td>4</td>
<td>1 k</td>
<td>3.952 V/k</td>
<td>0.648</td>
<td>$y^*V/k$</td>
<td>0.0337</td>
</tr>
<tr>
<td>5</td>
<td>1 k</td>
<td>4.905 V/k</td>
<td>0.635</td>
<td>$y^*V/k$</td>
<td>0.0344</td>
</tr>
<tr>
<td>6</td>
<td>1 k</td>
<td>5.937 V/k</td>
<td>0.688</td>
<td>$y^*V/k$</td>
<td>0.0329</td>
</tr>
<tr>
<td>7</td>
<td>1 k</td>
<td>6.937 V/k</td>
<td>0.607</td>
<td>$y^*V/k$</td>
<td>0.0339</td>
</tr>
<tr>
<td>8</td>
<td>1 k</td>
<td>7.900 V/k</td>
<td>0.642</td>
<td>$y^*V/k$</td>
<td>0.0352</td>
</tr>
<tr>
<td>9</td>
<td>1 k</td>
<td>8.820 V/k</td>
<td>0.614</td>
<td>$y^*V/k$</td>
<td>0.0368</td>
</tr>
<tr>
<td>10</td>
<td>1 k</td>
<td>9.693 V/k</td>
<td>0.582</td>
<td>$y^*V/k$</td>
<td>0.0388</td>
</tr>
<tr>
<td>11</td>
<td>1 k</td>
<td>10.616 V/k</td>
<td>0.615</td>
<td>$y^*V/k$</td>
<td>0.0382</td>
</tr>
<tr>
<td>12</td>
<td>1 k</td>
<td>11.473 V/k</td>
<td>0.537</td>
<td>$y^*V/k$</td>
<td>0.0411</td>
</tr>
<tr>
<td>13</td>
<td>1 k</td>
<td>12.259 V/k</td>
<td>0.524</td>
<td>$y^*V/k$</td>
<td>0.0449</td>
</tr>
<tr>
<td>14</td>
<td>1 k</td>
<td>12.967 V/k</td>
<td>0.472</td>
<td>$y^*V/k$</td>
<td>0.0498</td>
</tr>
<tr>
<td>15</td>
<td>1 k</td>
<td>13.592 V/k</td>
<td>0.417</td>
<td>$y^*V/k$</td>
<td>0.0564</td>
</tr>
<tr>
<td>16</td>
<td>1 k</td>
<td>14.204 V/k</td>
<td>0.408</td>
<td>$y^*V/k$</td>
<td>0.0602</td>
</tr>
<tr>
<td>17</td>
<td>1 k</td>
<td>14.708 V/k</td>
<td>0.336</td>
<td>$y^*V/k$</td>
<td>0.0732</td>
</tr>
<tr>
<td>18</td>
<td>1 k</td>
<td>15.095 V/k</td>
<td>0.259</td>
<td>$y^*V/k$</td>
<td>0.0951</td>
</tr>
<tr>
<td>19</td>
<td>1 k</td>
<td>15.361 V/k</td>
<td>0.177</td>
<td>$y^*V/k$</td>
<td>0.139</td>
</tr>
<tr>
<td>20</td>
<td>1 k</td>
<td>15.497 V/k</td>
<td>0.091</td>
<td>$y^*V/k$</td>
<td>0.271</td>
</tr>
</tbody>
</table>
As shown in Table 1, the maximum allowable displacement is $0.09354 \text{ m} = \left( \frac{1}{641} \right) h_t$, and the critical section occurs at the base of the shear wall.

8. COMPARISON WITH VARIOUS SELECTIVE SEISMIC CODES

In comparing the data obtained by equation (13) with results obtained by various seismic codes, it can be observed in Table 2 the following:

1- All codes do not take into consideration any variation in stiffness of a shear wall, nor they care as to weather the structure is relatively stiff or not.

2- All codes do not consider the geometry of the shear wall.

3- All codes do not consider the cracking limit of concrete and the yield limit of steel in a reinforced concrete section.

Table 2. Comparison of the formula suggested by the author and various selective seismic codes.

| Code                | Drift Range* | One Story Height =300 cm | 20 Story Building Constant Stiffness Height = 600 cm | 20 Story Building Variable Stiffness Example
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Drift Values (cm)</td>
<td>Drift Values (cm)</td>
<td>Drift Values (cm)</td>
<td>Drift Values (cm)</td>
</tr>
<tr>
<td>UBC91</td>
<td>0.004h - 0.005h</td>
<td>1.2 - 1.5</td>
<td>24 - 30</td>
<td>24. - 30.</td>
</tr>
<tr>
<td>UBC97</td>
<td>0.02 h - 0.25 h</td>
<td>6.0 - 7.5</td>
<td>120. - 150.</td>
<td>120. - 150.</td>
</tr>
<tr>
<td>IBC2006</td>
<td>0.01 h - 0.02 h</td>
<td>3.0 - 6.0</td>
<td>60. - 120.</td>
<td>60. - 120.</td>
</tr>
<tr>
<td>Lebanese Code</td>
<td>0.005h</td>
<td>0.15</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Algerian Code</td>
<td>0.0075h</td>
<td>2.25</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>French Code - PS92</td>
<td>0.0007h</td>
<td>0.21 - 0.36</td>
<td>4.2 - 7.2</td>
<td>4.2 - 7.2</td>
</tr>
<tr>
<td>Japanese Code</td>
<td>0.005h - 0.0083h</td>
<td>1.5 - 2.5</td>
<td>30. - 50.</td>
<td>30. - 50.</td>
</tr>
<tr>
<td>New Zealand Code</td>
<td>0.0067h - 0.010h</td>
<td>2.0 - 3.0</td>
<td>40. - 60.</td>
<td>40. - 60.</td>
</tr>
<tr>
<td>Khouri (Eqn. 13)</td>
<td>1.1</td>
<td>14.8</td>
<td>9.354</td>
<td></td>
</tr>
</tbody>
</table>

4. Some codes consider the zone to be an issue in the maximum allowable drift, while what we think is that, regardless of the zone, the structure has a displacement limit which if it is surpassed the structure will be in danger.

5. The formula of equation 13 suggested by the author in this article takes into consideration the structural stiffness, the shear wall geometry, the cracking limits of concrete and the yield limit of steel and the height of the building.

9. CONCLUSIONS

In this study, the finite element method along with structural dynamics and reinforced concrete design was used to analyze a shear building with variable lateral stiffness. The lateral stiffness matrix was assembled for variable stiffness in the stories, and the shear was obtained as a function of the displacement. A value for the displacement at any story was determined, and from which a function for the relative displacement between two stories was then obtained. Consequently, an equation for the maximum strain was resolved and a limiting value for the maximum allowable displacement within the elastic limits was obtained as a function of the height of a story, number of stories, depth of tension steel $d$ in a shear wall, the maximum allowable strain of steel $\varepsilon_{st}$ and the maximum allowable concrete strain $\varepsilon_c$. Note that shear building was analyzed like a beam with ignoring vertical loads and assuming constant lateral stiffness in all stories.

From the above examples one can conclude that the position of the critical section in the shear wall cannot be predicted before calculations because it depends on how the stiffness of the shear wall decreases when going upwards. In addition, the value of the maximum allowable drift at the top of the shear wall also depends on how the stiffness decreases with height.

This is so, keeping in mind that the values of the maximum allowable drift obtained in the above example can be increased if an allowable strain $\varepsilon_{st}$ larger than $\varepsilon_y$ is used which means that the yielding of steel is permitted.

What can be concluded from the comparison of the results obtained by the author and the other seismic codes is that when estimating drift one should check actual structural behavior rather than estimate the drift based on comfort of a person in upper floors or other factors that have nothing to do with the structural properties.

It is now left for the designing engineer to evaluate his structure and decide/choose a maximum allowable strain limit for concrete and for steel, and determine the corresponding maximum allowable drift values. Beyond these limits the designing engineer would know that the shear wall in question has passed the elastic limit in a shear building.
ACKNOWLEDGEMENTS

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