The vibrations of the elastic suspended rigid with harmonic seismic excitation

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Abstract: - The rigid solid elastically suspended is common to many technical accomplishments such as: buildings foundations placed on an elastic environment, thermal engines with plastic mounts, grabbing devices (of industrial robots) suspended on the compliance system etc. The elastic elements of these systems are tied to the base (mount, chassis). There is considered that the base has a vibratory movement of a rigid, movement that is constituted in seismic excitation and there are studied the vibrations that are produced in the elastically suspended solid system.

Keywords: - vibration, model, elastic supports, seismic

1. INTRODUCTION

In the previous work [6] it was determined the analytical expression of the excitation that appears when the base, at which are linked the springs of the hanged rigid, has a known motion.

In the present paper, considering that the base has a rigid body harmonic motion, we determine the vibrations induced in the elastic system of the suspended rigid and we make a numerical application.

2. NOTATIONS

We consider the rigid drawn in the figure 1 suspended by the spring generically denoted \( AB \), linked to the base and the notations:
- \( OXYZ \), the reference system of the rigid at the equilibrium, system considered as general reference frame;
- \( \delta \), the displacements of the point \( O \);
- \( \delta_X, \delta_Y, \delta_Z \), the projections of the vector \( \vec{\delta} \) onto the axes \( OX, OY, OZ \);
- \( \vec{\delta} \), the angular displacement (small) of the rigid;
- \( \theta_X, \theta_Y, \theta_Z \), the projection of the vector \( \vec{\theta} \) onto the axes \( OX, OY, OZ \);
- \( \vec{\theta} \), the angular displacement (small) of the base;
- \( \Delta_X, \Delta_Y, \Delta_Z \), the projections of the vector \( \vec{\Delta} \) onto the axes \( OX, OY, OZ \);
- \( \vec{\Delta} \), the displacement of the point \( O \) of the base;
- \( \vec{\psi} \), the angular displacement (small) of the base;
- \( \psi_X, \psi_Y, \psi_Z \), the projections of the vector \( \vec{\psi} \) onto the axes \( OX, OY, OZ \);

\[ s = L - \tilde{L} \]  

Figure 1. The rigid elastically suspended.
the matrices defined by the relations

\[
[r_a] = \begin{bmatrix}
0 & -Z_A & Y_A \\
Z_A & 0 & -X_A \\
-Y_A & X_A & 0 
\end{bmatrix}
\]

onto the axes \(OX, OY, OZ\);

\[
[r_c] = \begin{bmatrix}
0 & -Z_C & Y_C \\
Z_C & 0 & -X_C \\
-Y_C & X_C & 0 
\end{bmatrix}
\]

onto the axes \(OX, OY, OZ\);

\[
[J_0], \ [M], \text{ the matrix of the moments of inertia, respectively the matrix of inertia}
\]

\[
[J_0] = \begin{bmatrix}
J_X & -J_{XY} & -J_{XZ} \\
-J_{XY} & J_Y & -J_{YZ} \\
-J_{XZ} & -J_{YZ} & J_Z 
\end{bmatrix}
\]

\[
[M] = \begin{bmatrix}
m[I] & m[r_c] \\
m[r_c] & [J_0] 
\end{bmatrix}
\]

\[
[K_{AB}^{(1)}], \ [K_{AB}^{(2)}], \ [K_{AB}^{(3)}], \ [K_{AB}], \ [K_C], \ [K], \text{ the rigidity matrices}
\]

\[
K_{AB}^{(1)} = k \left(1 - \frac{s}{L}\right) [U]^T [U]
\]

\[
K_{AB}^{(2)} = \frac{ks}{L} [I] - [r_A] [r_A]^T
\]

\[
K_{AB}^{(3)} = -\frac{ks}{2} [0] [0] [r_A] [u] + [u] [r_A]
\]

\[
K_{AB} = K_{AB}^{(1)} + K_{AB}^{(2)} + K_{AB}^{(3)}
\]

\[
K_C = -\frac{mg}{2} [0] [0] [r_C] [u_C] + [u_C] [r_C]
\]

\[
[K] = K_C + \sum K_{AB};
\]

\[
K_{AB} = K_{AB}^{(1)} + K_{AB}^{(2)} + K_{AB}^{(3)}
\]

\[
K_{AB}^{(2)} = \frac{ks}{L} [I] - [r_B] [r_B]^T
\]

\[
K_{AB}^{(3)} = -\frac{ks}{2} [0] [0] [r_B] [u] + [u] [r_B]
\]

\[
K_{AB} = K_{AB}^{(1)} + K_{AB}^{(2)} + K_{AB}^{(3)}
\]
\[ [\bar{K}] = \sum [K_{AB}] \]  \hspace{2cm} (8)

3. THE DIFFERENTIAL METRICAL EQUATION OF THE VIBRATIONS

As was shown in paper [6] the column matrix of the excitation force, using the notations (3), (8), is given by the relation

\[ \{F_e\} = [\bar{K}]\{\Delta\} \]  \hspace{2cm} (9)

where \(\{\Delta\}\) is the column matrix of the base’s displacement, reduced at the point \(O\).

In the case when the base’s displacement is given by the linear displacement \(\Delta^*\) of the point \(O\) (fig. 1) and by the angular displacement \(\psi\) of the base, then one gets the relation

\[ \Delta = \Delta^* + O\bar{O}O \times \psi \]  \hspace{2cm} (10)

and, if we denote by \(\Delta_X^*, \Delta_Y^*, \Delta_Z^*, (X_0, Y_0, Z_0)\) the projections of the vectors \(\Delta^*, O\bar{O}O\) onto the axes \(OX, OY, OZ\) and by \(\{\Delta^*\}, \{r_O\}\) the matrices

\[ \{\Delta^*\} = \begin{bmatrix} \Delta_X^* \\ \Delta_Y^* \\ \Delta_Z^* \\ \psi_X \\ \psi_Y \\ \psi_Z \end{bmatrix} \]  \hspace{2cm} (11)

\[ \{r_O\} = \begin{bmatrix} 0 & -Z_O & Y_O \\ Z_O & 0 & -X_O \\ -Y_O & X_O & 0 \end{bmatrix} \]  \hspace{2cm} (12)

it results

\[ \{\Delta\} = \begin{bmatrix} [I] & [0] \end{bmatrix} \{\Delta^*\} \]  \hspace{2cm} (13)

In the case when the base’s vibrations are harmonic in the form

\[ \Delta = \overline{\Delta_O} \cos \omega t \]; \hspace{0.5cm} \psi = \overline{\psi_O} \cos \omega t \]  \hspace{2cm} (14)

then using the notations

\[ \{\Delta^*_O\} = \begin{bmatrix} \Delta_{OX}^* \\ \Delta_{OY}^* \\ \Delta_{OZ}^* \\ \psi_X \\ \psi_Y \\ \psi_Z \end{bmatrix} \]  \hspace{2cm} (15)

we deduce the equality

\[ \{\Delta_O\} = \begin{bmatrix} [I] & [0] \end{bmatrix} \{\Delta^*_O\} \]  \hspace{2cm} (16)

and the metrical expression of the excitation force is

\[ \{F_e\} = \begin{bmatrix} [K] \end{bmatrix} \{\Delta^*_O\} \cos \omega t \]  \hspace{2cm} (17)

Based on the relations established in the paper [6], one obtains the metrical equation of the vibrations for the elastically suspended rigid

\[ [M]\{\dot{\delta}\} + [K]\{\delta\} = [\bar{K}]\{\Delta^*_O\} \cos \omega t \]  \hspace{2cm} (18)

with the solution for the stationary regime

\[ \{\delta\} = [\bar{K}]^{-1} [K]\{\Delta^*_O\} \cos \omega t \]  \hspace{2cm} (19)

4. APPLICATION

Determine the answer at the harmonic excitation for the shell of mass \(m\) in figure 2 knowing that the point \(O\) of the base has the linear displacement \(tO\omega \psi^* \cos \cdot\) parallel to the axis \(OX\) and the rotational angular of the base is \(tO\omega \psi\) parallel to the axis \(OZ\).

Figure 2. The model of the shell elastically.

We consider that the springs \(AB, DE\) are identical with the stiffness equal to \(k = \frac{mg}{l}\), and at
the equilibrium the elongation of the spring \( FG \) of stiffness \( k \) is null.

From the equilibrium conditions it follows that the elongations of the springs \( AB \), \( CD \) are equal and denoting by \( s \) the common value, one obtains

\[
s = \frac{mg}{k \sqrt{2}} = \frac{l}{\sqrt{2}} \quad \text{and since} \quad L = 2l \sqrt{2} \quad \text{we deduce} \quad \frac{s}{L} = \frac{1}{4}.
\]

For the spring \( AB \) we successively deduce the relations

\[
[U] = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 2l \end{pmatrix}^T
\]

\[
[r_A] = \begin{bmatrix} 0 & 0 & l \\ 0 & 0 & -3l \\ -l & 3l & 0 \end{bmatrix}
\]

\[
[r_B] = \begin{bmatrix} 0 & 0 & 3l \\ 0 & 0 & -5l \\ -3l & 5l & 0 \end{bmatrix}
\]

\[
[K^{(1)}_{AB}] = \frac{3k}{8} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 2l \\ 1 & 1 & 0 & 0 & 0 & 2l \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2l & 2l & 0 & 0 & 0 & 4l^2 \end{bmatrix}
\]

\[
[K^{(2)}_{AB}] = \frac{k}{4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -l \\ 0 & 1 & 0 & 0 & 0 & 3l \\ 0 & 0 & 1 & l & -3l & 0 \\ 0 & 0 & l & 2l & -3l^2 & 0 \\ 0 & 0 & -3l & -3l^2 & 9l^2 & 0 \\ -l & 3l & 0 & 0 & 0 & 10l^2 \end{bmatrix}
\]

\[
[K^{(3)}_{AB}] = \frac{k}{4} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2l^2 & -4l^2 & 0 \\ 0 & 0 & -4l^2 & 6l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8l^2 \end{bmatrix}
\]

\[
[K^{(2)}_{AB}] = \frac{k}{4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -l \\ 0 & 1 & 0 & 0 & 0 & 3l \\ 0 & 0 & 1 & l & -3l & 0 \\ 0 & 0 & l & 2l & -3l^2 & 0 \\ 0 & 0 & -3l & -3l^2 & 9l^2 & 0 \\ -l & 3l & 0 & 0 & 0 & 10l^2 \end{bmatrix}
\]

\[
[K^{(3)}_{AB}] = \frac{k}{4} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2l^2 & -4l^2 & 0 \\ 0 & 0 & -4l^2 & 6l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8l^2 \end{bmatrix}
\]
We obtain the rigidity matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -3l \\
0 & 1 & 0 & 0 & 0 & -5l \\
0 & 0 & 1 & 3l & 5l & 0 \\
0 & 0 & l & 3l^2 & 5l^2 & 0 \\
0 & 0 & 3l & 9l^2 & 15l^2 & 0 \\
-1 & -3l & 0 & 0 & 0 & 18l^2
\end{bmatrix}
\]

and for the spring \( FG \) it results, \( s = 0 \)

\[
\{ U \} = (0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0)^T
\]

\[
\begin{bmatrix}
K^{(2)}_{DE} \\
K^{(2)}_{FG}
\end{bmatrix} = \begin{bmatrix} K^{(3)}_{DE} \\
K^{(3)}_{FG} \end{bmatrix} = [0];
\]

Knowing that between the linear displacements \( \overrightarrow{X}, \overrightarrow{X} \) at the points \( O, O_0 \) there exists the vector relation

\[
\overrightarrow{X} = \overrightarrow{X} + \mathbf{v} \times \overrightarrow{OO_0}
\]

one deduces the scalar relations \( \Delta_X = \Delta^* - 3l\psi \), \( \Delta_Y = \Delta_Z = 0 \), we obtain the relation of base’s displacements reduced at the point \( O \)

\[
\{ \Delta \} = (l \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)^T \psi_O \cos \omega t
\]

and the excitation reads

\[
\{ F_x \} = [K]\{ \Delta \} = \psi_O l \left( \frac{5}{4} \quad 0 \quad 0 \quad 0 \quad 12l \right)^T \cos \omega t.
\]

In this way one deduces the metrical equation of the vibrations with seismic harmonic excitation

\[
\begin{bmatrix}
m & 0 & 0 & 0 & 0 & 0 \\
0 & m & 0 & 0 & 0 & 0 \\
0 & 0 & m & 0 & 0 & 0 \\
0 & 0 & 0 & J_x & 0 & 0 \\
0 & 0 & 0 & 0 & J_y & 0 \\
0 & 0 & 0 & 0 & 0 & J_z
\end{bmatrix} \begin{bmatrix}
\ddot{\mathbf{r}}_p \end{bmatrix} = \frac{\psi_O l k}{4} \begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 4l \\
0 & 9 & 0 & 0 & 0 & 0 \\
0 & 0 & 9 & 0 & 0 & 0 \\
0 & 0 & 0 & 3l^2 & 0 & 0 \\
4l & 0 & 0 & 0 & 48l^2 & 0 \\
0 & 0 & 0 & 0 & 12l
\end{bmatrix} \begin{bmatrix}
\ddot{\mathbf{r}}_p \end{bmatrix}
\]

which separates in the scalar equations with excitation

\[
m \dddot{x} + \frac{5k}{4} \dddot{x} + kl \theta_x = \frac{5k}{4} \psi_O l \cos \omega t
\]

\[
J_z \dddot{\theta}_z + kl \ddot{\theta}_x + 12k \theta_x \theta_z + 3k \psi_O l^2 \cos \omega t
\]

and in the scalar equations without excitation

\[
m \dddot{y} + \frac{9k}{4} \dddot{y} = 0; m \dddot{z} + \frac{k}{2} \dddot{z} + k \theta_x = 0;
\]

\[
J_x \dddot{\theta}_x + \frac{kl}{2} \dddot{\theta}_z + 3k \dddot{\theta}_x \theta_z + 3k \theta_x \theta_z = 0; J_y \dddot{\theta}_y + \frac{15k l^2}{2} \dddot{\theta}_y = 0.
\]

The equations with excitation leads us to the solution in the stationary regime.
\[ \frac{\delta_x}{\theta_z} = \begin{bmatrix} -m\omega^2 + \frac{5k}{4} & kl \\ kl & -J_z\omega^2 + 12kl^2 \end{bmatrix} \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \end{bmatrix} k\psi_0 \cos \omega t . \]

For a numerical application we consider

\[ l = 0.2 \text{[m]} , \]
\[ m = 61.162 \text{[kg]} , \]
\[ k = 3000 \begin{bmatrix} \frac{N}{m} \\ \frac{m}{N} \end{bmatrix} , \]
\[ J_z = \frac{10}{3} ml^2 = 8.154 \text{[kgm}^2] , \]
\[ \psi_0 = 0.1 \text{[rad]} , \]
\[ \omega = 10 \text{[s}^{-1}] , \]

and it results

\[ \delta_x = -0.0137 \cos \omega t , \]
\[ \theta_z = 0.0708 \cos \omega t . \]

REFERENCES


