
Study of Vertical Non-linear Systems with Polynomial Characteristics

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Abstract: - In our paper we considered a vertical system of rigid bodies linked one to another and to the ground by non-linear springs. We obtained the differential equations of motion using the Lagrange second order equations and, based on them, we discussed the system of equations used to obtain the equilibrium positions. Considering two of the most used cases, name them the cubic non-linear springs, and the neo-Hookean non-linear springs, we deduced the conditions for the uniqueness of the equilibrium position. We also presented the type of the equilibrium and we studied the small oscillations around the stable equilibrium position. In our paper we studied the stability of the motion, too. The linear case is obtained as a particular case of those considered previous in this paper. Finally, we considered a numerical example, which describes the vertical vibrations of an automobile.

Keywords: - non-linear, polynomial, neo-Hookean, stability, vibrations

1. INTRODUCTION

In the literature of the field are studied the non-linear systems with one or two degrees of freedom. The non-linear systems with more degrees of freedom are, in general, avoided, because the mathematical complications those appear.

In the present paper we consider a system with n degrees of freedom, the masses being linked one to another and to the ground with non-linear elastic springs for which the elastic force has a polynomial expression.

2. WORKING MODEL

We consider a vertical system consisting of n masses linked one to another and to the ground by n non-linear springs (fig. 1). The masses of the n bodies are m_i , $i = \overline{1, n}$, and the elastic force in a spring is given by

$$F_j = k_j d + \varepsilon_j d^p, \quad j = \overline{1, n}, \quad (1)$$

where $k_j > 0$, d is the elongation of the spring, ε_j are real constants, and p is an integer, $p \neq 0$, $p \neq 1$. The n bodies can move only in the vertical direction, and therefore the system has n degrees of freedom.

3. THE EQUATIONS OF MOTION

Isolating the n masses (fig. 2), one obtains the equations of motion in the form

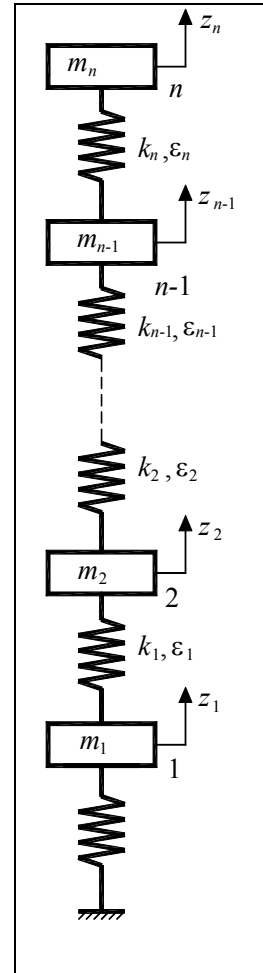


Figure 1. Mathematical model.

$$m_1 \ddot{z}_1 = k_2(z_2 - z_1) + \varepsilon_2(z_2 - z_1)^p - k_1 z_1 - \varepsilon_1 z_1^p - m_1 g, \quad (2a)$$

$$m_2 \ddot{z}_2 = k_3(z_3 - z_2) + \varepsilon_3(z_3 - z_2)^p - k_2(z_2 - z_1) - \varepsilon_2(z_2 - z_1)^p - m_2 g, \dots, \quad (2b)$$

$$m_i \ddot{z}_i = k_{i+1}(z_{i+1} - z_i) + \varepsilon_{i+1}(z_{i+1} - z_i)^p - k_i(z_i - z_{i-1}) - \varepsilon_i(z_i - z_{i-1})^p - m_i g, \dots, \quad (2c)$$

$$m_{n-1} \ddot{z}_{n-1} = k_n(z_n - z_{n-1}) + \varepsilon_n(z_n - z_{n-1})^p - k_{n-1}(z_{n-1} - z_{n-2}) - \varepsilon_{n-1}(z_{n-1} - z_{n-2})^p - m_{n-1} g, \quad (2d)$$

$$m_n \ddot{z}_n = -k_n(z_n - z_{n-1}) - \varepsilon_n(z_n - z_{n-1})^p - m_n g. \quad (2e)$$

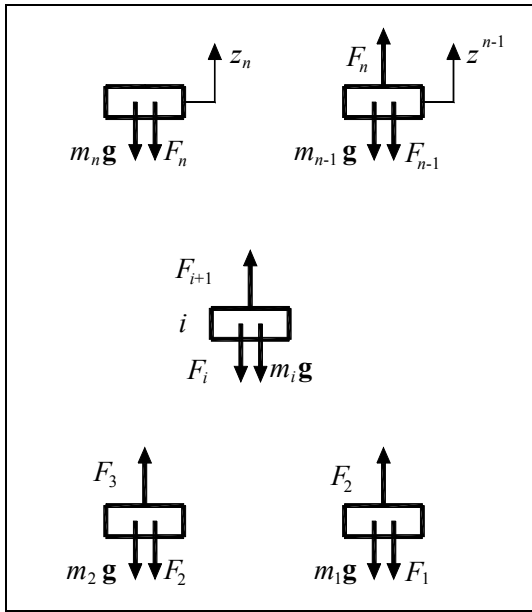


Figure 2. Isolation of the bodies

4. THE EQUILIBRIUM POSITIONS

These are obtained by vanishing the right-hand side terms of the equations (2)

$$k_2(z_2 - z_1) + \varepsilon_2(z_2 - z_1)^p - k_1 z_1 - \varepsilon_1 z_1^p - m_1 g = 0, \quad (3a)$$

$$k_3(z_3 - z_2) + \varepsilon_3(z_3 - z_2)^p - k_2(z_2 - z_1) - \varepsilon_2(z_2 - z_1)^p - m_2 g = 0, \dots, \quad (3b)$$

$$k_{i+1}(z_{i+1} - z_i) + \varepsilon_{i+1}(z_{i+1} - z_i)^p - k_i(z_i - z_{i-1}) - \varepsilon_i(z_i - z_{i-1})^p - m_i g = 0, \dots, \quad (3c)$$

$$k_n(z_n - z_{n-1}) + \varepsilon_n(z_n - z_{n-1})^p - k_{n-1}(z_{n-1} - z_{n-2}) - \varepsilon_{n-1}(z_{n-1} - z_{n-2})^p - m_{n-1} g = 0, \quad (3d)$$

$$-k_n(z_n - z_{n-1}) - \varepsilon_n(z_n - z_{n-1})^p - m_n g = 0. \quad (3e)$$

Summing the equations (3), it results

$$\varepsilon_1 z_1^p + k_1 z_1 + \sum_{i=1}^n m_i g = 0. \quad (4)$$

Let us assume, for the beginning, that p is a natural number, $p \geq 2$.

There exist two sub-cases:

– p is an even number. Making $z_1 \mapsto -z_1$ in the equation (4), it follows

$$\varepsilon_1 z_1^p - k_1 z_1 + \sum_{i=1}^n m_i g = 0. \quad (5)$$

If $\varepsilon_1 > 0$, then in the equation (4) there exists no variation of sign in the sequence of the coefficients, and in the equation (5) there exist two variations of sign. Applying the Descartes theorem, it results that the equation (4) has no real positive root and it has 0 or 2 real negative roots. If $\varepsilon_1 < 0$, then there exists a variation of sign in the sequence of the equation (4) and a variation of sign in the sequence of coefficients in equation (5). We obtain that the equation (4) has a real positive root and a negative real root;

– p is an odd number. Making $z_1 \mapsto -z_1$ in the equation (4), it results

$$-\varepsilon_1 z_1^p - k_1 z_1 + \sum_{i=1}^n m_i g = 0. \quad (6)$$

If $\varepsilon_1 > 0$, then the equation (4) has 0 real positive roots and 1 real negative root. If $\varepsilon_1 < 0$, then the equation (4) has a real positive root and 0 or 2 real negative roots.

Let us now consider the case when $p < 0$ and we denote $q = -p$, $q > 0$. The equation (4) reads now

$$k_1 z_1^{q+1} + \left(\sum_{i=1}^n m_i g \right) z_1^q + \varepsilon_1 = 0. \quad (7)$$

Again we have two sub-cases:

– p is an even number, $q+1$ is an odd number. Making $z_1 \mapsto -z_1$, the equation (7) transforms in

$$-k_1 z_1^{q+1} + \left(\sum_{i=1}^n m_i g \right) z_1^q + \varepsilon_1 = 0. \quad (8)$$

If $\varepsilon_1 > 0$, then the equation (7) has no real positive roots, but it has a real negative root. If $\varepsilon_1 < 0$, then the equation (7) has a real positive root and 0 or 2 real negative roots.

– p is an odd number, $q+1$ is an even number. Making $z_1 \mapsto -z_1$, the equation (7) becomes

$$k_1 z_1^{q+1} - \left(\sum_{i=1}^n m_i g \right) z_1^q + \varepsilon_1 = 0. \quad (9)$$

If $\varepsilon_1 > 0$, then the equation (7) has 0 real positive roots and 0 or 2 real negative roots. If $\varepsilon_1 < 0$, then the equation (7) has a real positive root and a real negative root.

It results from here that the equilibrium position is unique if $\varepsilon_1 > 0$ and p is a positive odd number or a negative even number.

From the equation (4) we find

$$\varepsilon_1 z_1^p + k_1 z_1 = - \left(\sum_{i=1}^n m_i g \right) \quad (10)$$

and from the first relation (3) with the notation

$$u = z_2 - z_1, \quad (11)$$

one deduces

$$\varepsilon_2 u^p + k_2 u + \left(\sum_{i=2}^n m_i g \right) = 0. \quad (12)$$

The discussion for the equation (12) is identical to those for the equation (4) with the observation that now the conclusion is for ε_2 . Step by step, one obtains that the equilibrium position is unique if $\varepsilon_i > 0$ for all $i = \overline{1, n}$ and p is a positive odd number or a negative even number.

Further on, we shall consider only these cases.

5. THE STABILITY OF THE EQUILIBRIUM

With the notations $\xi_i = z_i$, $\dot{\xi}_{i+n} = \dot{z}_i$, $i = \overline{1, n}$, the system (2) transforms in a system of $2n$ first order non-linear differential equations,

$$\dot{\xi}_1 = \xi_{n+1}, \dot{\xi}_2 = \xi_{n+2}, \dots, \dot{\xi}_n = \xi_{2n}, \quad (13)$$

$$\dot{\xi}_{n+1} = \frac{1}{m_1} \left[k_2 (\xi_2 - \xi_1) + \varepsilon_2 (\xi_2 - \xi_1)^p - k_1 \xi_1 - \varepsilon_1 \xi_1^p \right] - g, \quad \dots,$$

$$\dot{\xi}_{n+i} = \frac{1}{m_i} \left[k_{i+1} (\xi_{i+1} - \xi_i) + \varepsilon_{i+1} (\xi_{i+1} - \xi_i)^p - k_i (\xi_i - \xi_{i-1}) - \varepsilon_i (\xi_i - \xi_{i-1})^p \right] - g,$$

$$\dots,$$

$$\dot{\xi}_{2n} = \frac{1}{m_n} \left[-k_n (\xi_n - \xi_{n-1}) - \varepsilon_n (\xi_n - \xi_{n-1})^p \right] - g.$$

The Jacobi matrix reads

$$[J] = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \\ j_{n+1,1} & j_{n+1,2} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ j_{n+2,1} & j_{n+2,2} & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & j_{n+3,2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & j_{2n-1,2n-1} & j_{2n-1,2n} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & j_{2n,2n-1} & j_{2n,2n} & 0 & 0 & \dots & 0 \end{bmatrix} \quad (14)$$

and the characteristic equation has the form

$$\begin{vmatrix} j_{n+1,1} - \lambda^2 & j_{n+1,2} & \dots & 0 \\ j_{n+2,1} & j_{n+2,2} - \lambda^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & j_{2n-1,2n} \\ 0 & 0 & \dots & j_{2n,2n} - \lambda^2 \end{vmatrix} = 0. \quad (15)$$

We shall consider now the case $n = 2$. The characteristic equation (15) becomes now

$$\begin{vmatrix} j_{31} - \lambda^2 & j_{32} \\ j_{41} & j_{42} - \lambda^2 \end{vmatrix} = 0 \quad (16)$$

or, equivalently,

$$\lambda^4 - (j_{31} + j_{42})\lambda^2 + j_{31}j_{42} - j_{32}j_{41} = 0. \quad (17)$$

Since the system (13) has the form

$$\dot{\xi}_1 = \xi_3, \dot{\xi}_2 = \xi_4, \quad (18)$$

$$\begin{aligned}\dot{\xi}_3 &= \frac{1}{m_1} \left[k_2 (\xi_2 - \xi_1) + \varepsilon_2 (\xi_2 - \xi_1)^p - \right. \\ &\quad \left. - k_1 \xi_1^p - \varepsilon_1 \xi_1^p \right] - g, \\ \dot{\xi}_4 &= \frac{1}{m_2} \left[-k_2 (\xi_2 - \xi_1) - \varepsilon_2 (\xi_2 - \xi_1)^p \right] - g,\end{aligned}$$

it immediately results

$$\begin{aligned}j_{31} &= -\frac{k_2}{m_1} - \frac{\varepsilon_2 p}{m_1} (\xi_2 - \xi_1)^{p-1} - \\ &\quad - \frac{k_1}{m_1} - \frac{\varepsilon_1 p}{m_1} \xi_1^{p-1},\end{aligned}\quad (19)$$

$$j_{32} = \frac{k_2}{m_1} + \frac{\varepsilon_2 p}{m_1} (\xi_2 - \xi_1)^{p-1}, \quad (20)$$

$$j_{41} = \frac{k_2}{m_2} + \frac{\varepsilon_2 p}{m_2} (\xi_2 - \xi_1)^{p-1}, \quad (21)$$

$$j_{42} = -\frac{k_2}{m_2} - \frac{\varepsilon_2 p}{m_2} (\xi_2 - \xi_1)^{p-1}. \quad (22)$$

Let us now consider that $p > 0$, p is an odd number, and $\varepsilon_i > 0$, $i = 1, 2$.

From the formulas (19)–(22) we find

$$\begin{aligned}j_{31} < 0, \quad j_{32} > 0, \quad j_{41} > 0, \quad j_{42} < 0, \\ j_{31}j_{42} - j_{32}j_{41} > 0\end{aligned}\quad (23)$$

and therefore the equation (17) has pure imaginary roots; the equilibrium is stable, but not asymptotically stable.

If $p < 0$, p is an even number, and $\varepsilon_i > 0$, $i = 1, 2$, keeping into account that at equilibrium $\xi_1 < 0$, $\xi_2 < 0$, we again obtain the relations (23) such that the equilibrium is stable, but not asymptotically stable.

6. THE STABILITY OF THE MOTION

We shall take $n = 2$.

We shall limit ourselves at the most known cases.

If $p = 3$, let us make $\xi_i \mapsto \xi_i + \zeta_i$, $i = \overline{1, 4}$, where ζ_i are the deviations, sufficiently small. From the system (18) we get

$$\dot{\zeta}_1 = \zeta_3, \quad \dot{\zeta}_2 = \zeta_4, \quad (24)$$

$$\begin{aligned}\dot{\zeta}_3 &= \frac{1}{m_1} \left[k_2 (\zeta_2 - \zeta_1) - k_1 \zeta_1 + \right. \\ &\quad \left. + 3\varepsilon_2 (\xi_2 - \xi_1)^2 (\zeta_2 - \zeta_1) + \right. \\ &\quad \left. + 3\varepsilon_2 (\xi_2 - \xi_1) (\zeta_2 - \zeta_1)^2 + \varepsilon_2 (\zeta_2 - \zeta_1)^3 - \right. \\ &\quad \left. - 3\varepsilon_1 \xi_1^2 \zeta_1 - 3\varepsilon_1 \xi_1 \zeta_1^2 - \zeta_1^3 \right], \\ \dot{\zeta}_4 &= \frac{1}{m_2} \left[-k_2 (\zeta_2 - \zeta_1) - \right. \\ &\quad \left. - 3\varepsilon_2 (\xi_2 - \xi_1)^2 (\zeta_2 - \zeta_1) - \right. \\ &\quad \left. - 3\varepsilon_2 (\xi_2 - \xi_1) (\zeta_2 - \zeta_1)^2 - \varepsilon_2 (\zeta_2 - \zeta_1)^3 \right]\end{aligned}$$

or, by linearization,

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_3, \quad \dot{\zeta}_2 = \zeta_4, \\ \dot{\zeta}_3 &= \frac{1}{m_1} \left[-k_2 - k_1 - \right. \\ &\quad \left. - 3\varepsilon_2 (\xi_2 - \xi_1)^2 (\zeta_2 - \zeta_1) - 3\varepsilon_1 \xi_1^2 \right] \zeta_1 + \\ &\quad + \frac{1}{m_1} \left[k_2 + 3\varepsilon_2 (\xi_2 - \xi_1)^2 \right] \zeta_2, \\ \dot{\zeta}_4 &= \frac{1}{m_2} \left[k_2 + 3\varepsilon_2 (\xi_2 - \xi_1)^2 \right] \zeta_1 + \\ &\quad + \left[-k_2 - 3\varepsilon_2 (\xi_2 - \xi_1)^2 \right] \zeta_2.\end{aligned}\quad (25)$$

The system (25) can be put in the form

$$\begin{aligned}\ddot{\zeta}_1 + a_{11}\zeta_1 - a_{12}\zeta_2 &= 0, \\ \ddot{\zeta}_2 - a_{21}\zeta_1 + a_{22}\zeta_2 &= 0,\end{aligned}\quad (26)$$

where $a_{ij} > 0$, $i, j = \overline{1, 2}$, the notations being obvious.

From the second relation (26) it follows

$$\begin{aligned}\zeta_1 &= \frac{1}{a_{21}} (\ddot{\zeta}_2 + a_{22}\zeta_2), \\ \ddot{\zeta}_1 &= \frac{1}{a_{21}} (\zeta_2^{(iv)} + a_{22}\ddot{\zeta}_2), \quad a_{21} \neq 0,\end{aligned}\quad (27)$$

such that one obtains the equation

$$\zeta_2^{(iv)} + (a_{11} + a_{22})\ddot{\zeta}_2 + (a_{11}a_{22} - a_{12}a_{21})\zeta_2 = 0 \quad (28)$$

and since

$$\begin{aligned}a_{11} + a_{22} &> 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0, \\ (a_{11} - a_{22})^2 + 4a_{12}a_{21} &> 0\end{aligned}\quad (29)$$

it results that the roots of the characteristic equation of the equation (28) are pure imaginary, and the solution of the system (26) writes as harmonic vibrations.

If $p = -2$, then the system in deviations reads in the form

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_3, \quad \dot{\zeta}_2 = \zeta_4, \\ \dot{\zeta}_3 &= \frac{1}{m_1} \left[k_2(\zeta_2 - \zeta_1) - k_1\zeta_1 - \right. \\ &\quad \left. - \frac{2\varepsilon_2}{(\xi_2 - \xi_1)^3} (\zeta_2 - \zeta_1) + \dots \right], \\ \dot{\zeta}_4 &= \frac{1}{m_2} \left[-k_2(\zeta_2 - \zeta_1) + \right. \\ &\quad \left. + \frac{2\varepsilon_2}{(\xi_2 - \xi_1)^3} (\zeta_2 - \zeta_1) + \dots \right]\end{aligned}\quad (30)$$

or, by linearization,

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_3, \quad \dot{\zeta}_2 = \zeta_4, \\ \dot{\zeta}_3 &= \\ &= \frac{1}{m_1} \left[-k_2 - k_1 + \frac{2\varepsilon_2}{(\xi_2 - \xi_1)^3} \right] \zeta_1 + \\ &\quad + \frac{1}{m_2} \left[k_2 - \frac{2\varepsilon_2}{(\xi_2 - \xi_1)^3} \right] \zeta_2, \\ \dot{\zeta}_4 &= \\ &= \frac{1}{m_2} \left[k_2 - \frac{2\varepsilon_2}{(\xi_2 - \xi_1)^3} \right] \zeta_1 + \\ &\quad + \frac{1}{m_2} \left[-k_2 + \frac{2\varepsilon_2}{(\xi_2 - \xi_1)^3} \right] \zeta_2.\end{aligned}\quad (31)$$

The system (21) can be brought to the form (26), but the discussion is more complicate. Let us observe that a sufficient condition to obtain harmonic motions is given by

$$\xi_2 < \xi_1, \quad (32)$$

to verify the relations $a_{ij} > 0$, $i, j = \overline{1, 2}$, and (29).

7. APPLICATION

Let us consider the case $p = 3$ and the numerical values

$$\begin{aligned}m_1 &= 100 \text{ kg}, \quad m_2 = 1300 \text{ kg}, \\ k_1 &= 1.4 \cdot 10^8 \text{ N/m}, \quad k_2 = 70000 \text{ N/m}, \\ \varepsilon_1 &= 10^4 \text{ N/m}^3, \quad \varepsilon_2 = 10^2 \text{ N/m}^3, \\ g &= 10 \text{ m/s}^2\end{aligned}\quad (33)$$

for which result the equations

$$10^4 z_1^3 + 1.4 \cdot 10^6 z_1 + 14000 = 0, \quad (34)$$

$$10^2 (z_2 - z_1)^3 + 70000(z_2 - z_1) + 13000 = 0, \quad (35)$$

with the solution

$$z_1 = -0.01 \text{ m}, \quad z_2 = -0.185 \text{ m}. \quad (36)$$

As we discussed, the equilibrium position and the motion are stable, but not asymptotically stable.

8. CONCLUSION

In our paper we presented the general vertical system with non-linear elastic springs. For this kind of systems we studied the stability of the equilibrium and of the motion.

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