On the Anomalous Doppler Effect

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Abstract: When an object is moving along an elastic guide the Doppler waves are excited. For a static observer, the Doppler wave emitted by the object coming to observer has the frequency higher than the frequency of the object. For the case when the object speed exceeds the minimum waves velocity the anomalous Doppler wave is excited. This time, the wave has the frequency lower than the frequency of the object. The anomalous Doppler waves may generate the instability of the vibration of the moving object. This paper presents the normal and anomalous Doppler effect for the case of a moving force along an infinite Euler-Bernoulli beam. This issue is relevant for the case of modern trains travelling along a track with soft soil when the trains speed exceeds the phase velocity of the waves induced in the track, and the results derived from a numerical simulation of this aspect are presented.

Keywords: anomalous Doppler wave, instability, limit cycle, track, railway vehicle

1. INTRODUCTION

The dynamic forces that develop at the wheel-rail contact are moving forces, travelling at the same velocity as the vehicle.

The rail and the track, in its entirety, are continuous elastic mediums where any perturbation travels at a certain velocity.

There are bending waves in the rail, and their propagation speed is much higher than the usual train speeds; the same thing happens with the high speed trains. But this situation changes when it comes to the rail embankment. There are many types of waves propagating through the rail embankment: longitudinal, transversal and Raleigh, which are surface waves (fig. 1) [1]. The Raleigh waves have the lowest propagation speed. When the soils are soft (peat, clay, marine clay, etc), the speed of these waves is even lower and reaches the speed range of the high speed trains or even much lower. It is possible that the train speed exceeds the speed of the elastic waves, induced by their circulation, and this leads to an increase in the track stress by rising the deformation under the loads coming from the wheels [2, 3]. Likewise, the Doppler effect takes place, which is a general phenomenon for a load to travel through an elastic medium.

When the propagation speed of the elastic waves is exceeded, the anomalous Doppler waves occur, which results into the accumulation of the wheel-rail system energy, thus contributing to the instability phenomenon of the wheel travelling on the rail [4-6].

Figure 1. Types of waves: (a) longitudinal; (b) transversal; (c) Rayleigh.
This paper presents a range of issues that are linked to how the Doppler waves propagate through an infinite Euler-Bernoulli beam; as for application regarding the unstable vibration behaviour, the case of an oscillator travelling on a beam with a viscoelastic foundation is being examined.

2. DOPPLER EFFECT

It is considered as one-dimensional elastic medium where a progressive harmonic wave propagates, with the angular frequency \( \omega \) and wave number \( \beta \). Travelling of an abscissa point \( x \) at the time \( t \) shows in the equation

\[ w(x,t) = w_0 \cos(\omega t - \beta x), \quad (1) \]

where \( w_0 \) is the amplitude of the wave.

Should a point travels at the speed \( V \), then it will be at \( x = Vt \) at time \( t \), and its travelling is described by the equation

\[ z(t) = w_0 \cos(\omega t - \beta Vt) = w_0 \cos\Omega t. \quad (2) \]

It means that the point has a periodic movement, with an angular frequency different than the wave’s, namely the fixed abscissa point’s. The relation between the two angular frequencies is kinematic in nature and may be written as below

\[ \omega = \beta V + \Omega. \quad (3) \]

This relation is known under the name of \textit{kinematic invariant}.

The issue may be also looked at with reversed terms. This would be the angular frequency compared to a fixed point of a wave sent out through an elastic medium by a perturbation of an angular frequency \( \Omega \) travelling at the speed \( V \).

Since the wave propagates at the speed \( c = \omega/\beta \), we have

\[ \omega = \frac{\Omega}{1 - V/c}. \quad (4) \]

out of the kinematic invariant equation.

This equation illustrates the Doppler effect for the one-dimensional case and proves that the frequency perceived by a fixed observer is higher than the frequency of the moving source at a certain speed towards the observer.

Should the perturbation moves away from the observer, then the speed changes its sign and the fraction denominator becomes higher than unit

\[ \omega = \frac{\Omega}{1 + V/c} \quad (5) \]

and the frequency perceived by the fixed observer is smaller than the frequency of the perturbation.

The Doppler effect was first explained by \textit{Christian Doppler} in 1842. In fact, the Doppler effect does not happen only for the elastic waves, but it shows for any type of perturbation that propagates through a medium at a certain speed and is observed from two points moving towards each other. More, the Doppler effect is present when the medium itself moves. The frequency difference that a fixed observer sees when, for example, the perturbation source moves toward him; this is what explains the shorter time when he perceives that perturbation as a result of shortening the distance between him and the source, while the perturbation travels the distance between the source and the observer at a constant speed.

When the speed of the source exceeds the speed of propagation, the equation (4) is not valid any longer since the angular frequency becomes negative for the observer, which physically does not make sense. Therefore, the equation must be changed into

\[ \omega = \frac{\Omega}{|1 - V/c|}. \quad (6) \]

It may be noticed that for \( V > 2c \), the angular frequency perceived by the fixed observer to which the source is going towards is smaller than the source’s – and this is the essence of the anomalous Doppler effect. The anomalous Doppler effect may take to the destabilization of the source movement due to the negative damping effect. The waves that propagate under these circumstances are called anomalous Doppler waves. Unlike them, the normal Doppler waves have a higher frequency than the source generating them when they recede from the observer.

3. THE DOPPLER WAVES PROPAGATION ALONG A BEAM

We have a uniform infinite Euler-Bernoulli beam, with the mass per unit length \( m \), the bending moment \( EI \), where \( E \) is the longitudinal elasticity module, and \( I \) is the inertia moment of the transversal section. The beam is under the action of a harmonic force that travels at the constant speed \( V \) along it, so that at the moment \( t = 0 \) goes through the origin of the reference frame (fig. 2).
The beam movement equation may be written as

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = Q \delta(x - Vt) \cos \omega t,$$  

where $\omega$ is the force angular frequency, and $\delta(.)$ is Dirac’s delta function.

The boundary conditions attached to the equation (7) express the fact that the beam vibration has finite amplitude, or zero, at the infinite boundaries

$$0 \leq \lim_{x \to \pm\infty} w(x,t) \leq M.$$

It is preferable working with a moving reference frame related by the force. Thus, we have the variable change

$$x = x_1 + Vt.$$

We have the following derivates

$$\frac{\partial^n w}{\partial x^n} \rightarrow \frac{\partial^n w}{\partial x_1^n}, \quad \frac{\partial^n w}{\partial t^n} \rightarrow \left( \frac{\partial w}{\partial t} - V \frac{\partial w}{\partial x_1} \right)^n.$$

Upon introducing the derivates above into the movement equation, this will become

$$EI \frac{\partial^4 w}{\partial x_1^4} + mV^2 \frac{\partial^2 w}{\partial x_1^2} - 2mV \frac{\partial^2 w}{\partial x_1 \partial t} + m \frac{\partial^2 w}{\partial t^2} = Q \cos \omega t \delta(x_1)$$

The boundary conditions will have the same behaviour

$$0 \leq \lim_{x_1 \to \pm\infty} w(x_1,t) \leq M.$$

A question may be asked on what requirement needs to be met so as this equation admit a harmonic solution like

$$w(x_1,t) = \bar{w}(x_1) \cos \omega t,$$

or expressed as a complex variable,

$$\bar{w}(x_1,t) = \bar{w}(x_1) \exp i\omega t.$$

where $\bar{w}(x_1)$ is the beam amplitude, and $i^2 = -1$.

The result is the below equation

$$\frac{d^4 \bar{w}(x_1)}{dx_1^4} + mV^2 \frac{d^2 \bar{w}(x_1)}{dx_1^2} - 2i\omega \bar{w}(x_1) \frac{d\bar{w}(x_1)}{dx_1} = 0$$

Upon having in the homogenous equation

$$\frac{d^4 \bar{w}(x_1)}{dx_1^4} + mV^2 \frac{d^2 \bar{w}(x_1)}{dx_1^2} - 2i\omega \bar{w}(x_1) \frac{d\bar{w}(x_1)}{dx_1} - m\Omega^2 \bar{w}(x_1) = 0$$

solutions like this

$$\bar{w}(x_1) = \bar{w} \exp \gamma x_1,$$

we will have the characteristic equation

$$EI \frac{d^4 \bar{w}(x_1)}{dx_1^4} + mV^2 \frac{d^2 \bar{w}(x_1)}{dx_1^2} - 2i\omega \bar{w}(x_1) \frac{d\bar{w}(x_1)}{dx_1} - m\Omega^2 \bar{w}(x_1) = 0$$

where $p=\sqrt{EIm}/\Omega$.

Finally, the solutions of the characteristic equations are obtained

$$\gamma_{1,3} = \pm \sqrt{\pm(V^2 + 4p\Omega)/2p}$$

or

$$\gamma_{2,4} = \frac{iV \pm \sqrt{4p\Omega - V^2}}{2p}.$$

To have the beam response, the Green function method may be applied, associated to the equation operator (15). This function is being built by considering the boundary conditions and its properties related to continuity. The Green function will be as follows

$$G^-(x_1, \xi) = \frac{1}{EI} \left\{ \exp(\gamma_1(x_1 - \xi)) + \exp(\gamma_2(x_1 - \xi)) \prod_{k=1}^{2} (\gamma_k - \gamma_1) \prod_{k=2}^{3} (\gamma_k - \gamma_2) \right\},$$

or $-\infty < x_1 < \xi$. 

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**Figure 2.** Euler-Bernoulli beam under a mobile harmonic load.
The beam displacement in any section $x$ compared to the static referential is calculated by going back to the old variable.

The results of a numerical application are presented further down, when considering the following data: $EI=6.42$ MN/m and $m=1800$ kg/m. Figure 3 shows the displacement of a beam point situated at 10 m to the reference origin while a harmonic force with a unit amplitude and frequency of 20 Hz is travelling at a speed of 35 m/s.

The diagram includes the displacement of the moving point on the beam opposite the harmonic force. The effect of the normal Doppler waves is being noticed. The frequency is higher when the force comes closer to the reference point, then it is smaller. Likewise, when the calculation point reaches behind the force, the displacement amplitude is higher, as it is in the shock wave of the moving force. Should the force travels at a higher speed than the group’s, then it will produce anomalous Doppler waves when reaching the observation point, and these waves has a lower frequency than the frequency of the force generating it.

This issue is presented in figure 4, where the harmonic force is still 20 Hz but moves at a 208 m/s speed, while the observation point is at 100 m to the reference frame. After the force has passed by the observation point, it will manifest in a higher frequency wave that decreases while speed increases.
4. THE INSTABILITY OF A MOVING OSCILLATOR ON A BEAM ON A VISCOELASTIC FOUNDATION

The explanation of the vibration instability of a moving body travelling along an elastic structure has been given by Metrikine [7], who proved that the anomalous Doppler waves increase the energy of the transversal oscillations by the negative damping effect. It should be underlined that the generation of the anomalous Doppler waves is only the prerequisite for instability but not sufficient, as instability is the result of the interaction between the moving body and the elastic structure.

The issue of instability for various types of moving sub-systems on elastic structures modelling the vehicle-track system has been dealt with in many papers [8-10]. Their main purpose was to identify the instability areas in the space of the system parameters, by using the so-called method of D decomposition. Nevertheless, the dynamic behaviour of the unstable motion, when the sub-system speed is higher than the critical speed has not been yet studied. There are still two exceptions, i.e. our papers, published in Journal of Sound and Vibration and in Nonlinear Dynamics [11, 12].

The case of a moving oscillator along an Euler-Bernoulli beam on a viscoelastic foundation has been considered. (fig. 5). The moving oscillator has two masses related by a Kelvin-Voigt system. This model may be looked at one of the simplest vehicle travelling on the track. In fact, the low mass of the oscillator represents the wheel, and the upper mass of the oscillator is the suspended mass of the vehicle. Also, the Kelvin-Voigt system is the vehicle suspension. This model serves to highlight the basic properties of the stability loss phenomenon, from the quality perspective. The values of the model parameter are: \( m = 2500 \text{ kg/m} \), \( EI = 4.76 \text{ MNm}^2 \), \( k = 2 \text{ MN/m}^2 \), \( c = 65.24 \text{ kNs/m}^2 \), \( M_1 = 2000 \text{ kg} \), \( M_2 = 1425 \text{ kg} \), \( k_0 = 1.974 \text{ MN/m} \), \( c_0 = 25.133 \text{ kNs/m} \), \( Q_0 = 80 \text{ kN} \), \( C_H = 96.38 \text{ GN/m}^{3/2} \). It has to be mentioned that is the static load \( Q_0 \) and \( C_H \) is the Hertz’s constant. The Hertz’s constant depends on the radius of the wheel/rail profiles.

When oscillation becomes unstable, the dynamic behaviour has two components: the transient and the steady-state behaviour, as seen in figure 6, where the contact force at the 172 m/s speed is shown, a value that is within the instability range. At the beginning of the transient behaviour, the wheel and the track are in a permanent contact, but the amplitude rises on a continuous basis. At the end, the contact is lost – and this is indicated by the zero values of the contact force – and the dynamic behaviour is characterized by an exponential increase of the contact force. Finally, the movement gets stabilized and takes the shape of a periodical movement – the steady-state behaviour.

The detailed picture of the steady-state behaviour is presented in figure 7. It has to be mentioned that the positive sign means a downward movement, and the negative one is upward, thus corresponding to the sense of the reference frame. The system movement is generated by the action of two opposite factors: the anomalous Doppler waves pushing the wheel upwards and the static load pushing the wheel towards the track.

The system vibration is a limit cycle with a frequency lower than the transient behaviour’s, when the wheel and rail are in a permanent contact. This issue may be explained by the fact that the wheel movement is slower during its detachment off the rail. On the impact, the force has a very high peak, 18 times the static load (fig. 8).

![Figure 5. Moving oscillator on the Euler-Bernoulli beam with viscoelastic foundation.](image-url)
Figure 6. Contact force at the 172 m/s speed.

Figure 7. The limit cycle at 172 m/s: —, the beam displacement at the contact point; – – –, wheel displacement; ····, the displacement of the suspended mass.

Figure 8. The contact force during the limit cycle at 172 m/s.
4. CONCLUSIONS

Elastic waves propagate through the track, due to the contact force of wheel-rail. Their propagation speed is lower than for the tracks built on soft soils, so that the speed range of the modern trains is being reached. In such situation, the anomalous Doppler waves occur, with a smaller frequency than the source generating them when propagating towards a fixed point of observation. The emergence of the anomalous Doppler waves represents the prerequisite of the wheel-rail vibrations instability occurrence.

The paper herein shows how the Doppler waves turn up, modelling after an infinite Euler-Bernoulli beam. The solution of the movement equation is obtained via the Green function of the differential operator of the movement equation.

The normal and anomalous Doppler wave’s emergence is highlighted. The instability due to the anomalous Doppler effect is showcased for the case of a moving oscillator on an Euler-Bernoulli beam on a viscoelastic foundation. The oscillator instability takes the shape of a limit cycle. This is the outcome of the action of anomalous Doppler waves and of the static load. For this reason, the limit cycle becomes a succession of shocks of extremely high amplitude.

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REFERENCES