
A Method for Single-Tone Frequency Estimation

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Abstract: - The present paper addresses the estimation of the frequency of a single-tone sine wave signal corrupted by additive white Gaussian noise. Brief background information is given about the signal model and the Maximum Likelihood Estimation (MLE) method. Further the newly proposed method is described, involving two-step coarse and fine search of the frequency index that maximized the signal modified periodogram, based on MLE and Weighted Averaging Estimation, respectively. Computer simulations are performed to compare the mean-square-error of the proposed estimator with that of some other frequency estimators along with Cramér-Rao lower bound.

Keywords: - single-tone, frequency estimation, method.

1. INTRODUCTION

The estimation of the parameters (amplitude, frequency and phase) of a single-tone sine wave signal is a classic problem in the signal processing and measurement practice. It is important in the measurement of the system amplitude and phase responses, impedance response, phase difference, total harmonic distortions, intermodulation distortions, etc.; in the fields of instrumentation and measurement, communication, radar and sonar systems, acoustics and vibration, biomedicine, astronomy, etc. In some cases only one or two (e.g. amplitude and frequency) of the above mentioned parameters are of interest, hence a different approaches can be applied to resolve the problem.

An overview of the different methods for determining the parameters of sine wave signals are given in [1, 2]. Mainly Discrete Fourier Transform (DFT) based techniques are used to measure the frequency of the signal of interest. The measurement procedure involves a data acquisition, signal digitizing and windowing of the signal samples and computing of the DFT magnitude spectrum or periodogram.

Further processing often includes a Maximum Likelihood Estimation (MLE) technique [3]. Various improvements of this method are given in [3-7], based on the finer estimation of the MLE result via different interpolation techniques. In this paper we present yet another method for single-tone frequency

estimation using a two-step procedure: the classic coarse MLE and the finer Weighted-Averaging Estimation (WAE).

2. BACKGROUND

2.1. Signal model

Let's consider a real-world single-tone sine wave analog signal (mathematically real, time limited, noise embedded), defined as [2]

$$x(t) = X_m \sin(2\pi f_x t + \varphi_x) + z(t) \quad (1)$$

where: X_m is a amplitude of the signal;

f_x – frequency of the signal;

t – time;

φ_x – initial phase of the signal;

$z(t)$ – i.i.d, zero mean, additive white Gaussian noise $AWGN(0, \sigma^2)$.

The signal is measured via some kind of data acquisition (DAQ) device or system and digitalized with sampling frequency f_s , according Nyquist–Shannon sampling theorem

$$x[n] = X_m \sin(2\pi f_x n T_s + \varphi_x) + z[n], \quad (2)$$

where: $n = \{1, \dots, N\} \in \mathbb{Z}$ is the sample number;

$N \in \mathbb{Z}$ – number of all samples;
 T_s – sampling time interval.

The unknown frequency of the signal is the object of estimation.

2.2. Maximum Likelihood Estimation

Let's calculate the one-sided modified periodogram of the sampled and weighed with window function $w[n]$ signal $x[n]$ as [8]

$$\dot{X}[k] = \frac{2}{f_s \sum_{n=1}^N w[n]^2} \left| \sum_{n=1}^N x[n] w[n] e^{-j \frac{2\pi kn}{N}} \right|^2, \quad (3)$$

where $k \in \mathbb{Z}$ is the number of the spectral component.

In (3) the normalization factor could be omitted only if the frequency estimation is a must in order to reduce the computation complexity. The considerations about the window function selection will be given further in the paper.

The DFT in (3) is calculated at a discrete linear frequency points given with [9]

$$f[k] = k \Delta f = k \frac{f_s}{N}. \quad (4)$$

Then the frequency of the signal of interest could be approximated as [9]

$$\hat{f}_x \cong f[\hat{k}_m], \quad (5)$$

where

$$\hat{k}_m = \arg \max_{k \in \mathbb{Z}} (\dot{X}[k]) \quad (6)$$

is the MLE of k – the value of k that maximizes the periodogram $\dot{X}[k]$ [3]. One must note that according to (5) the correct estimation of the frequency index is of fair significance, since it affects the exact signal frequency estimation (measurement).

3. PROPOSED METHOD DESCRIPTION

3.1. Motivation and method description

When the signal frequency is estimated using MLE (6), the largest error occurs when the frequency is located in the middle between two DFT bins, (i.e. the error depends of the frequency

resolution Δf of the DFT). The maximum relative error is [9]

$$\varepsilon_{\max} = \frac{f_s}{2Nf_x}. \quad (7)$$

The analysis of (7) leads to the following suggestions: (i) the sampling frequency must be kept as low as possible, at the edge of the Nyquist–Shannon criterion; (ii) the number of the acquired samples must be as high as possible. When the frequency error is still not sufficient, a finer estimation of the periodogram index k_m must be applied.

We proposed a modified numeric MLE-based estimation technique named DFT-MLE-WAE, involving two-step coarse and fine search (cf. Fig. 1).

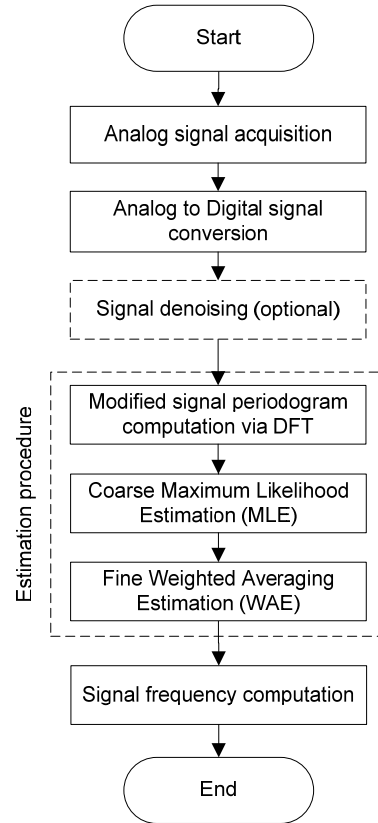


Figure 1. The algorithm of the proposed frequency estimation procedure. The estimation of the signal frequency is obtained via estimation of the frequency index that maximizes the periodogram using classic Maximum Likelihood Estimation and finer Weighted Averaging Estimation

The algorithm has three basic steps:

1) Computation of the modified periodogram of the sampled analogue signal, according (3);

2) MLE-based coarse search for the frequency index $\hat{k}_m$, that maximizes the periodogram via (6);

3) Finer search via WAE for the value of frequency index $\tilde{k}_m$, so a better estimation of the signal frequency can be made.

3.2. Weighted averaging

Weighted averaging (WA) (a.k.a. weighted arithmetic mean) is an estimation technique [10], a more general case of the simple arithmetic mean. Vis-a-vis the arithmetic mean in the WA, every data sample has a different contribution to the final result, since in order for each sample to be averaged it is assigned a weight – the higher weighted coefficient, the higher the relative importance of the sample.

We proposed a finer estimation of the frequency index $\hat{k}_m$ to be made using the periodogram samples X_{mi} assigned as the weighted coefficients to the corresponding frequency indexes k , as follows

$$\tilde{k}_m = \frac{\sum_{i=-p}^p X_m[\hat{k}_m + i] \cdot (\hat{k}_m + i)}{\sum_{i=-p}^p X_m[\hat{k}_m + i]}, \quad (8)$$

where: $\tilde{k}_m \in \mathbb{R}$ is the index of the finally estimated spectral component;

p – the number of the frequency indexes in the $\hat{k}_m$'s range that will contribute to the $\tilde{k}_m$ estimation.

We propose that all frequency indexes k situated in the main-lobe of the window should contribute in the estimation (*cf.* Fig. 2), hence the limit p depends of the main-lobe width (MLW) of the window function used in (3)

$$p = \left\lfloor \frac{MLW}{2} \right\rfloor, \quad (9)$$

where $\lfloor \bullet \rfloor$ means rounding towards zero.

If the periodogram is interpolated by factor of I (i.e. the number of DFT points are $N' = N \cdot I$), then p must be increased by the same factor I and rounded again towards zero. In general better results are achieved when relatively wide windows are used, since more samples contribute in the WAE. The list of some suitable windows and their parameters of interest are shown in Table 1 [11, 12].

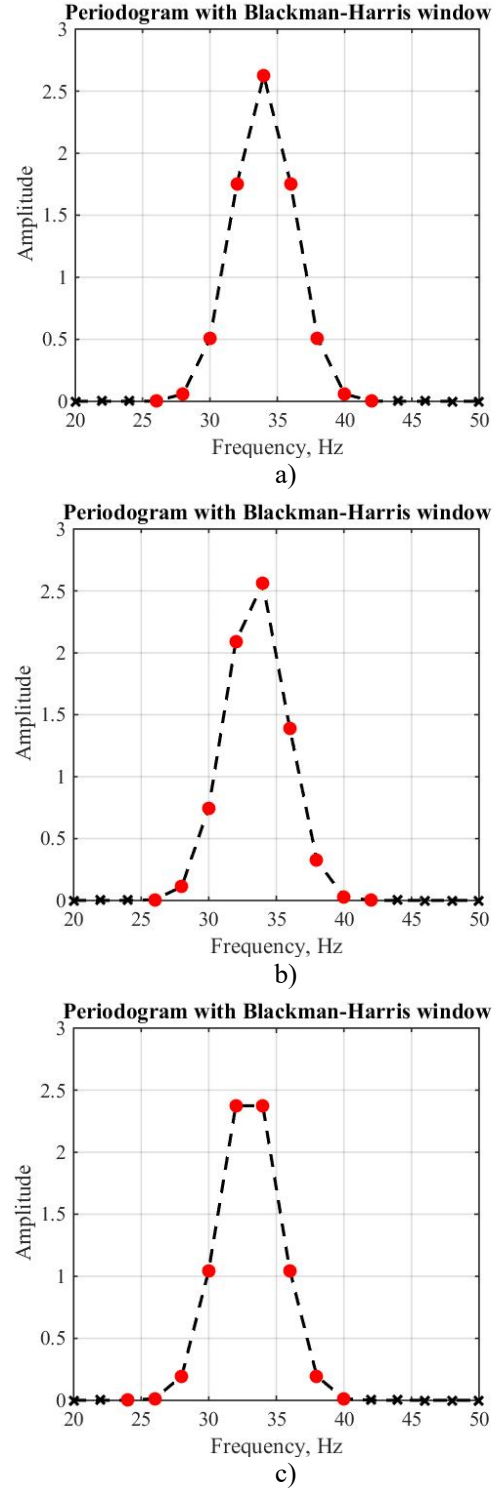


Figure 2. Three modified periodograms ($f_s = 512$ Hz, $N = 256$, $\Delta f = 2$ Hz) of single-tone sine wave signals with frequencies $f_0 = \{34, 33.5, 33\}$ Hz shown on a), b) and c) respectively and implementation of a Blackman-Harris window. The DFT-points in red are the ones that contribute in the finer Weighted Averaging Estimation

Table 1. Parameters of interest of some suitable window functions

Window function	Main-lobe width, bins	Relative Side-lobe attenuation, dB
Blackman-Harris (4-term)	8	-92
Dolph-Chebyshev ($\alpha = 5$)	8	-100
Nuttall (minimum 4-term)	8	-92
Kaiser ($\alpha = 4$)	8.24	-94.4

Once the value of $\tilde{k}_m$ is known the frequency of the measured signal could be approximated as

$$\tilde{f}_x \cong f[\tilde{k}_m]. \quad (10)$$

Since we used several spectral points (typically 4) on the left and the right side of the spectral peak, the last must lie in the interval $\hat{k}_m = \{p+1, \dots, N-(p+1)\}$.

4. SIMULATION RESULTS

The mean-square-error (MSE) of the predictor (or the estimator in general) of the frequency, obtained through M observations, is defined as [1, 10]

$$\begin{aligned} MSE(\tilde{f}_x) &\equiv \sigma_{\tilde{f}_x}^2 = \mathbb{E}\left\{\left(\tilde{f}_x - f_x\right)^2\right\} \\ &\approx \frac{1}{M} \sum_{i=1}^M \left(\tilde{f}_{xi} - f_{xi}\right)^2 \end{aligned} \quad (11)$$

The Cramér-Rao lower bound (CRLB) of the MSE of the frequency estimation for the signal model (2) is given with [2]

$$CRLB \equiv \sigma_{\tilde{f}_{x \min}}^2 = \frac{12}{4\pi^2 N(N^2 - 1)} 10^{-SNR/10}, \quad (12)$$

where the signal-to-noise ratio (SNR) of (1) is [1]

$$SNR = 10 \log_{10} \left(\frac{X_m^2}{2\sigma^2} \right). \quad (13)$$

In order to compare the variance of the frequency error of different types of estimators we are considering the log-efficiency [2]

$$e_{\log} = \log_{10} \left(\frac{MSE}{CRLB} \right). \quad (14)$$

Computer simulations were performed using Matlab® to examine the performance of the proposed estimator. A data sequences with length of $N = 256$ samples was used at sampling frequency $f_s = 512$ Hz and signal frequency $f_0 = 137.365$ Hz and the MSE was determined using $M = 1000$ trials at SNR values from 0 to 150 dB, spaced at 1 dB (cf. Fig. 3).

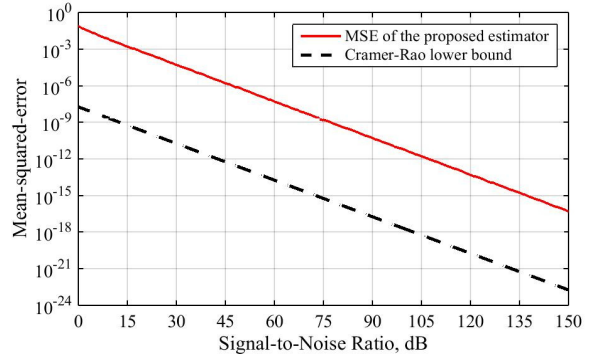


Figure 3. Simulation results for the Mean-squared-error of the proposed frequency estimator and the corresponding Cramér-Rao Lower Bound vs. Signal-to-Noise ratio. The parameters of the simulation are $f_0 = 137.365$ Hz, $f_s = 512$ Hz, $N = 256$, $\Delta f = 2$ Hz, Blackman-Harris window

Finally the log-efficiency is calculated vs. SNR (cf. Fig. 4) and the obtained result is compared with other type of frequency estimators given at [2] (cf. Table 2). The simulations were repeated using all of the window functions given in Table 1, with different signal frequencies and with zero-padded DFT and the obtained results are virtually the same as shown in Fig. 4.

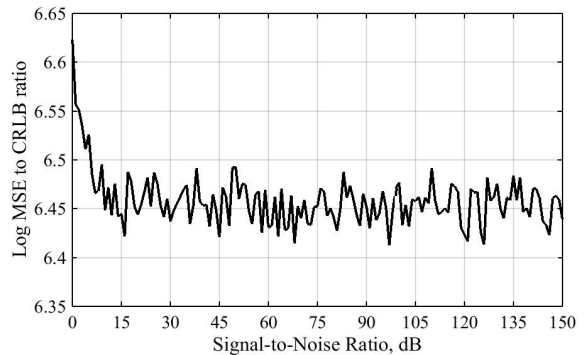


Figure 4. Simulation results for the logarithmic Mean-squared-error to Cramér-Rao Lower Bound ratio vs. Signal-to-Noise ratio of the proposed frequency estimator. The parameters of the simulation are $f_0 = 137.365$ Hz, $f_s = 512$ Hz, $N = 256$, $\Delta f = 2$ Hz, Blackman-Harris window

Table 2. The log-efficiency of the proposed and some others frequency estimators

Estimator	$e_{\log}$
Classic DFT-MLE estimator	7.57
Phase derivative estimator	7.41
Proposed DFT-MLE-WAE estimator	6.45
Grandke estimator	6.12
Phase difference estimator	6.10
Phase reassignment estimator	6.10
ESPRIT estimator	0.71

5. CONCLUSIONS

In the paper we presented a new single-tone sine wave frequency estimator abbreviated DFT-MLE-WAE. The estimator is DFT-based using a two-step procedure of fine and a coarse search for the frequency index k_m – the argument that maximizes the modified periodogram of the signal. The coarse search is obtained using classic MLE, and the fine search is realized via WAE. The method is tested and the average MSE for 1000 trials is calculated for different SNR values (*cf.* Fig. 3). Further the obtained value is compared with the MSE of another known frequency estimators (*cf.* Table 2). One can inference that the proposed DFT-MLE-WAE estimator has a smaller MSE of the well-known classic DFT-MLE and phase derivative estimators, but bigger than MSE the others. However the proposed estimator is yet another method that can be used in frequency measurement practice and it has a new impact in the signal parameters estimation theory. The possible application of this contribution is where the exact determining of the signal frequency is a must, including but not limited to acoustic measurements, vibration analysis, radar and sonar systems, biomedicine.

The method is implemented in Matlab® software environment as a Matlab-script [13] and could be

used for frequency estimation of any type of stationary periodic signals (e.g. sine wave, rectangular, triangular, etc.).

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